

Fundamentos de álgebra y cálculo

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Las preguntas a continuación están relacionadas con conceptos matemáticos básicos necesarios para el curso de Crecimiento Económico. Intenta primero responder estas preguntas por tu cuenta. Luego puedes consultar las respuestas en la segunda parte del documento para comprobar tu trabajo. Para quienes tengan dificultades con algunos temas, recomiendo revisar los siguientes temas en *Sydsæter and Hammond*, “*Essential mathematics for economic analysis*” (SH) o *Larson and Edward*, “*Cálculo*” (LE) los cuales puedes encontrar en nuestra biblioteca:

- Trabajar con funciones exponenciales (Capítulo 4.9 (SH), 5.4 (LE))
- Trabajar con funciones logarítmicas (Capítulo 4.10 (SH), 5.1 (LE))
- Funciones inversas (Capítulo 5.3 (SH)+(LE))
- Diferenciación (Capítulo 6 (SH), 2 (LE))
- Optimización (Capítulo 9 (SH), 3 (LE))
- Integración (Capítulo 10.1-10.3 (SH), 4 (LE))

1. Exponents

Simplify the following expressions

- (a) $(3x^2y^2)^2$
- (b) x^0/x^{-2}
- (c) $(3x^{-2})(2x^3)^2$

2. Logarithmic functions

Rewrite the following as functions of $\ln(x)$ and $\ln(y)$:

- (a) $\ln(x^5 \frac{y}{3})$
- (b) $\ln(x^{\frac{1}{3}} y^{-\frac{2}{3}})$

3. Derivatives

For each of the following functions, $y = f(x)$, provide the first and the second derivative.

- (a) $y = f(x) = 4 + 2x^2$
- (b) $y = f(x) = 2x^{\frac{1}{2}}$
- (c) $y = f(x) = \frac{1}{2} \ln(2x + 1)$

4. Graphs of functions

For each of the following functions, $y = f(x)$, find the limits of the functions and provide a plot of the graph.

(a) $y = f(x) = x + x^{-1}$

(b) $y = f(x) = -4 - 2x + x^2$

(c) $y = f(x) = 2 + \ln x$

5. Inverse functions

For each of the following functions, $y = f(x)$, find the inverse function $f^{-1}(y)$.

(a) $y = f(x) = 3 + x^2$

(b) $y = f(x) = 5 - x$

(c) $y = f(x) = 3e^{2x}$

6. Solving equations

For each of the following equations, find the solution(s) or indicate whether no solution exists

(a) $-x + 2x^2 = -4$

(b) $2x + x^2 + 1 = 0$

(c) $\frac{1}{2} \ln x + 1 = 1$

7. Systems of equations

For each of the following systems of equations, find the solution(s) or indicate whether no solution exists

(a) $x + y = 5, 2x - y = 1$

(b) $2x - 4y = 6, 2y + x = 3$

(c) $x + 2y = 4, 2y = 7 - x$

8. Partial derivatives

For each of the following functions, $z = f(x, y)$, provide first and the second derivatives with respect to x and with respect to y .

(a) $z = f(x, y) = x^\alpha y^\beta$

(b) $z = f(x, y) = \alpha \ln x + \beta \ln y$

(c) $z = f(x, y) = x + \alpha \ln(\beta y + 1)$

9. Global maxima/minima

For each of the following functions, $y = f(x)$, determine whether it attains a global maximum or minimum. If they do, solve for (x^*, y^*) at the critical points.

(a) $y = f(x) = 3 - 2x^2 + x$

(b) $y = f(x) = \frac{1}{2}x^2 - x + 1$

(c) $y = f(x) = -1 + 3x$

Answers

1. Exponents

(a) $(3x^2y)^3 = 27x^6y^3$

(b) $x^0/x^{-2} = x^2$

(c) $(3x)^{-2}(2x^3)^2 = \frac{4}{9}x^4$

2. Logarithmic functions

(a) $5\ln(x) + \frac{1}{3}\ln(y)$

(b) $\frac{1}{3}\ln(x) - \frac{2}{3}\ln(y)$

3. Derivatives

(a) $y = 4 + 2x^2$

$$y' = \frac{d}{dx}(4 + 2x^2) = 0 + 4x = 4x$$

$$y'' = \frac{d}{dx}(4x) = 4$$

(b) $y = 2x^{1/2}$

$$y' = 2 \cdot \frac{1}{2}x^{-1/2} = x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$y'' = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$$

(c) $y = \frac{1}{2}\ln(2x + 1)$

$$y' = \frac{1}{2} \cdot \frac{1}{2x + 1} \cdot 2 = \frac{1}{2x + 1}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{2x + 1}\right) = -\frac{2}{(2x + 1)^2}$$

4. Graphs of functions

Limits

(a)

$$x \mapsto -\infty, y \mapsto -\infty$$

$$x \mapsto \infty, y \mapsto \infty$$

$$x_- \mapsto 0, y \mapsto -\infty$$

$$x_+ \mapsto 0, y \mapsto \infty$$

(b)

$$x \mapsto -\infty, y \mapsto \infty$$

$$x \mapsto \infty, y \mapsto \infty$$

(c)

$$x_+ \mapsto 0, y \mapsto -\infty$$

$$x \mapsto \infty, y \mapsto \infty$$

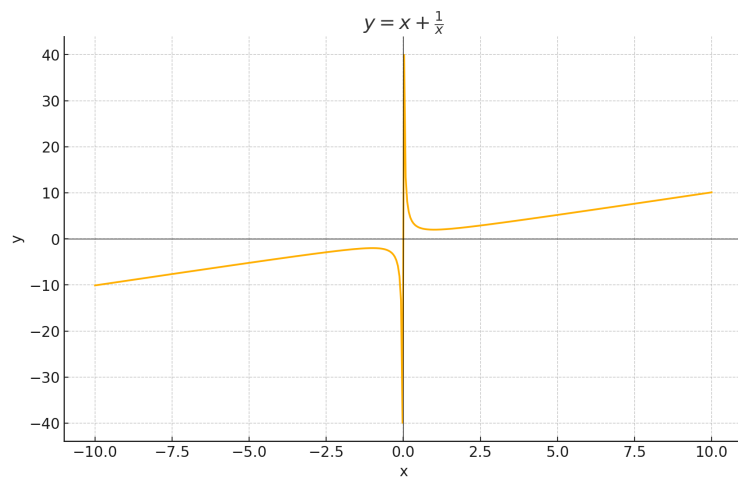


Figure 1: (a) $y = x + x^{-1}$

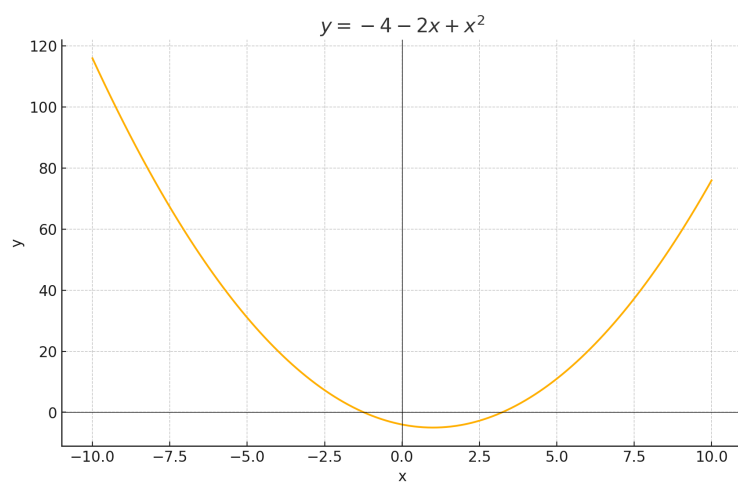


Figure 2: (b) $y = -4 - 2x + x^2$

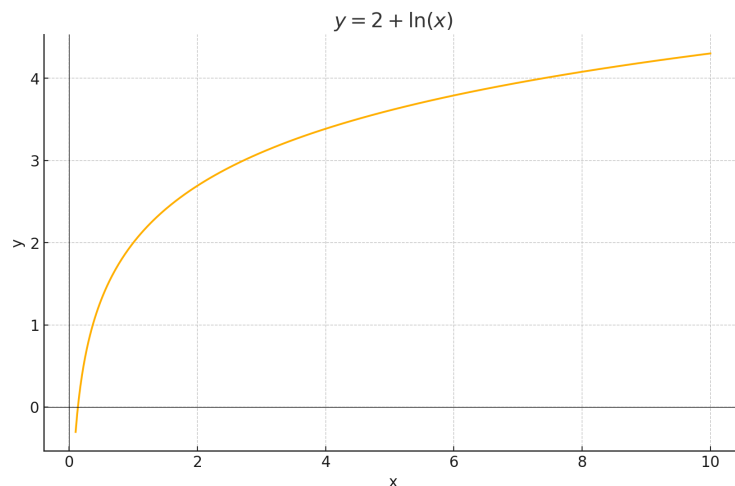


Figure 3: (c) $y = 2 + \ln x$

5. Inverse functions

(a) $y = 3 + x^2$

Restrict domain to $x \geq 0$:

$$y = 3 + x^2$$

$$y - 3 = x^2$$

$$x = \sqrt{y - 3}$$

$$f^{-1}(y) = \sqrt{y - 3}$$

(b) $y = 5 - x$

$$y = 5 - x$$

$$x = 5 - y$$

$$f^{-1}(y) = 5 - y$$

(c) $y = 3e^{2x}$

$$y = 3e^{2x}$$

$$\frac{y}{3} = e^{2x}$$

$$\ln\left(\frac{y}{3}\right) = 2x$$

$$x = \frac{1}{2} \ln\left(\frac{y}{3}\right)$$

$$f^{-1}(y) = \frac{1}{2} \ln\left(\frac{y}{3}\right)$$

6. Solving equations

(a) $-x + 2x^2 = -4$

$$2x^2 - x + 4 = 0$$

Use quadratic formula:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2} = \frac{1 \pm \sqrt{-31}}{4} \Rightarrow \text{No real solution}$$

(b) $2x + x^2 + 1 = 0$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0 \Rightarrow x = -1$$

(c) $\frac{1}{2} \ln x + 1 = 1$

$$\frac{1}{2} \ln x = 0 \Rightarrow \ln x = 0 \Rightarrow x = 1$$

7. Systems of equations

(a)

$$x + y = 5$$

$$2x - y = 1$$

Add: $3x = 6 \Rightarrow x = 2$

$$y = 5 - 2 = 3$$

(b)

$$2x - 4y = 6$$

$$x - 2y = 3 \quad (\text{Divide 1st by 2nd})$$

Same equation $\Rightarrow$ Infinite solutions

(c)

$$x + 2y = 4$$

$$2y = 7 - x \Rightarrow x + 2y = x + 2y = 4 \text{ and } x + 2y = 7 \Rightarrow \text{No solution}$$

8. Partial derivatives

(a) $z = f(x, y) = x^\alpha y^\beta$

First Partial Derivatives

Partial with respect to x: Use the power rule while treating y^β as constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x^\alpha y^\beta) = y^\beta \cdot \frac{d}{dx} (x^\alpha) = y^\beta \cdot \alpha x^{\alpha-1}$$

Partial with respect to y: Similarly, treat x^α as constant and differentiate y^β :

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^\alpha y^\beta) = x^\alpha \cdot \frac{d}{dy} (y^\beta) = x^\alpha \cdot \beta y^{\beta-1}$$

Second Partial Derivatives

Second with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} (\alpha x^{\alpha-1} y^\beta) = y^\beta \cdot \alpha(\alpha-1) x^{\alpha-2}$$

Mixed partial:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (\beta x^\alpha y^{\beta-1}) = \alpha \beta x^{\alpha-1} y^{\beta-1} \quad (1)$$

Second with respect to y:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} (\beta x^\alpha y^{\beta-1}) = x^\alpha \cdot \beta(\beta-1) y^{\beta-2}$$

(b) $z = f(x, y) = \alpha \ln x + \beta \ln y$

First Partial Derivatives

Partial with respect to x: Use the derivative of $\ln x$:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (\alpha \ln x + \beta \ln y) = \alpha \cdot \frac{1}{x}$$

Partial with respect to y: Use the derivative of $\ln y$, treating $\ln x$ as constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (\alpha \ln x + \beta \ln y) = \beta \cdot \frac{1}{y}$$

Second Partial Derivatives

Second with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\alpha \cdot \frac{1}{x} \right) = -\alpha \cdot \frac{1}{x^2}$$

Mixed partial:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\beta}{y} \right) = 0 \quad (2)$$

Second with respect to y:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\beta \cdot \frac{1}{y} \right) = -\beta \cdot \frac{1}{y^2}$$

(c) $z = f(x, y) = x + \alpha \ln(\beta y + 1)$

First Partial Derivatives

Partial with respect to x: Only the term x depends on x , the rest is treated as constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x + \alpha \ln(\beta y + 1)) = 1$$

Partial with respect to y: Apply the chain rule to differentiate $\ln(\beta y + 1)$:

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (\alpha \ln(\beta y + 1)) = \alpha \cdot \frac{1}{\beta y + 1} \cdot \frac{d}{dy}(\beta y + 1) \\ &= \alpha \cdot \frac{1}{\beta y + 1} \cdot \beta = \frac{\alpha \beta}{\beta y + 1} \end{aligned}$$

Second Partial Derivatives

Second with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (1) = 0$$

Mixed partial:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\alpha \beta}{\beta y + 1} \right) = 0 \quad (3)$$

Second with respect to y: Apply the quotient rule or differentiate the result using the chain rule:

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{d}{dy} \left(\frac{\alpha \beta}{\beta y + 1} \right) = \alpha \beta \cdot \frac{d}{dy} \left(\frac{1}{\beta y + 1} \right) \\ &= \alpha \beta \cdot \left(-\frac{\beta}{(\beta y + 1)^2} \right) \\ &= \frac{-\alpha \beta^2}{(\beta y + 1)^2} \end{aligned}$$

9. Global maxima/minima

(a) $y = 3 - 2x^2 + x$

$$y' = -4x + 1 = 0 \Rightarrow x = \frac{1}{4}$$

$$y = 3 - 2\left(\frac{1}{4}\right)^2 + \frac{1}{4} = 3 - \frac{1}{8} + \frac{1}{4} = \frac{27}{8}$$

$$\text{Max at } \left(\frac{1}{4}, \frac{27}{8}\right)$$

(b) $y = \frac{1}{2}x^2 - x + 1$

$$y' = x - 1 = 0 \Rightarrow x = 1$$

$$y = \frac{1}{2}(1)^2 - 1 + 1 = \frac{1}{2}$$

$$\text{Min at } \left(1, \frac{1}{2}\right)$$

(c) $y = -1 + 3x$

Linear function: no max or min unless domain is restricted.